FILTRATIONS OF SIMPLICIAL FUNCTORS AND THE NOVIKOV CONJECTURE

C. Ogle

The Ohio State University

Oct., 2004; revised Feb. 2005

Abstract. We show that the Strong Novikov Conjecture for the maximal C^*-algebra $C^*(\pi)$ of a discrete group π is equivalent to a statement in topological K-theory for which the corresponding statement in algebraic K-theory is always true. We also show that for any group π, rational injectivity of the full assembly map for $K_*^t(C^*(\pi))$ follows from rational injectivity of the restricted assembly map.

Contents

Key words and phrases. simplicial resolution, filtrations, assembly map, Novikov Conjecture.

0 Introduction

Let π be a countable discrete group. The Novikov conjecture for π is equivalent to the assertion that the assembly map for the Witt groups of $\mathbb{Z}[\pi]$,

$$H_*(B\pi; \underline{\underline{W}}(\mathbb{Z})) \to W_*(\mathbb{Z}[\pi]) \tag{0.1}$$

is rationally injective (here $H_*(B\pi; \underline{\underline{W}}(\mathbb{Z}))$ denotes the homology of $B\pi$ with coefficients in the spectrum $\underline{\underline{W}}(\mathbb{Z})$). The Novikov Conjecture states the Novikov Conjecture for π is true for all discrete groups π [N]. Let $C^*(\pi)$ denote the maximal C^* algebra of π. As with Witt theory, there is an assembly map

$$KU_*(B\pi) = H_*(B\pi; \underline{\underline{K}}^t(\mathbb{C})) \to K_*^t(C^*(\pi)) \tag{0.2}$$

where $\underline{\underline{K}}^t$ denotes topological K-theory spectrum, and $K_*^t(\_)$ its homotopy groups. The Strong Novikov Conjecture (SNC) for π asserts that the above assembly map for $C^*(\pi)$ is rationally injective [Ka]. The Strong Novikov Conjecture is the statement that SNC is true for all discrete groups π. Variants and generalizations of these conjectures exist for both topological and algebraic K-theory ([BC], [BCH], [C1], [C2], [DL], [FJ], [GV], [HR], [Y1]), and have been verified for many types of discrete groups ([Ka], [Mi], [KS], [BHM], [CM], [CGM], [Y2]).

After tensoring with $\mathbb{Q}$, the assembly maps in both (0.1) and (0.2) can be alternatively written as

$$\begin{gathered} \oplus_{n\in\mathbb{Z}} H_{*-4n}(B\pi;\mathbb{Q}) \to W_*(\mathbb{Z}[\pi])\otimes\mathbb{Q} \\ \oplus_{n\in\mathbb{Z}} H_{*-2n}(B\pi;\mathbb{Q}) \to K_*^t(C^*(\pi))\otimes\mathbb{Q} \end{gathered} \tag{0.3}$$

In both cases, there is a <u>connective assembly map</u>

$$\begin{gathered} \oplus_{n\in\mathbb{N}} H_{*-4n}(B\pi;\mathbb{Q}) \to W_*(\mathbb{Z}[\pi])\otimes\mathbb{Q} \\ ku_*(B\pi)\otimes\mathbb{Q} = \oplus_{n\in\mathbb{N}} H_{*-2n}(B\pi;\mathbb{Q}) \to K_*^t(C^*(\pi))\otimes\mathbb{Q} \end{gathered} \tag{0.4}$$

given by the restriction of the assembly maps in (0.3) to summands indexed by $n \geq 0$, and a <u>restricted assembly map</u>

$$\begin{gathered} H_*(B\pi;\mathbb{Q}) \to W_*(\mathbb{Z}[\pi])\otimes\mathbb{Q} \\ H_*(B\pi;\mathbb{Q}) \to K_*^t(C^*(\pi))\otimes\mathbb{Q} \end{gathered} \tag{0.5}$$

given by the further restriction to the summand corresponding to $n = 0$. From the periodicity isomorphisms $W_*(R) \cong W_{*+4}(R)$ and $K^t_*(A) \cong K^t_{*+2}(A)$ (R a discrete ring with involution, A a complex Banach algebra), one easily sees that the assembly map of either (0.1) or (0.2) is rationally injective if and only if its corresponding connective assembly map is rationally injective. However, it is not obvious that one can restrict any further. Our first result is (cf. Corollary 4.5):

Theorem A. *For a given discrete group π the assembly map in (0.2) is rationally injective if and only if the corresponding restricted assembly map in (0.5) is injective.*

Although we do not give an explicit proof in this paper, the techniques used in verifying Theorem A hold equally well for $W_*(\mathbb{Z}[\pi])$, yielding the analogue of Theorem A for the Witt groups of $\mathbb{Z}[\pi]$.

Our second result consists of i) a reformulation of SNC in topological K-theory, and ii) a proof of the algebraic K-theoretic analogue of the reduction in i). Given an augmented free simplicial resolution $\Gamma.^+$ of a discrete group π, we define a sequence of $\mathbb{C}$-algebras $\widetilde{C}^i_{n-1}(\Gamma.^+)$, $n \geq 1$, abelian groups A_n, and maps

$$A_n \twoheadrightarrow H_n(B\pi;\mathbb{Q})$$

$$A_n \to K^a_1(\widetilde{C}^i_{n-1}(\Gamma.^+)) \otimes \mathbb{Q} \to K^t_1(\widetilde{C}^i_{n-1}(\Gamma.^+)) \otimes \mathbb{Q}$$

where $K^a_1(\widetilde{C}^i_{n-1}(\Gamma.^+))$ resp. $K^t_1(\widetilde{C}^i_{n-1}(\Gamma.^+))$ denotes the the first algebraic resp. topological K-group of $\widetilde{C}^i_{n-1}(\Gamma.^+)$, equipped with the fine topology, and the second map is induced by the natural transformation from algebraic to topological K-theory.

Statement (A,n) Each $y \in A_n$ mapping to a non-zero element under the projection to $H_n(B\pi;\mathbb{Q})$ maps to a non-zero element in $K^a_1(\widetilde{C}^i_{n-1}(\Gamma.^+)) \otimes \mathbb{Q}$.

Statement (T,n) Each $y \in A_n$ mapping to a non-zero element under the projection to $H_n(B\pi;\mathbb{Q})$ maps to a non-zero element in $K^t_1(\widetilde{C}^i_{n-1}(\Gamma.^+)) \otimes \mathbb{Q}$.

Theorem B. *Statement (A,n) is true for all $n \geq 1$. Moreover, the SNC is true for π if and only if Statement (T,n) is true for all $n \geq 1$.*

The truth of either statement does not depend on choice of free resolution. Statement (A,n) is true for all free simplicial resolutions, and if Statement (T,n) is true for one free simplicial resolution of π, it is true for all others.

The paper consists of six sections. In section one, we prove some relevant properties of simplicial C^*-algebras, and show how they can be used to define a filtration on π_0 (which is a C^*-algebra). In section two we provide an alternative method for filtering the homotopy (resp. homology) groups of $F(-1)$, where F is an augmented simplicial object in a suitable category of spectra, spaces or chain complexes. In section three we identify the filtration in a basic case, used later in section five. In section four, we show that two apriori distinct filtrations defined on $K_*^t(C^*(\pi))$, using sections one and two, in fact agree. In section 5, we use this to prove Theorem 1. Section six contains some additional results on the C^i-algebras introduced in section five.

This paper is based on an approach to the Novikov conjecture investigated by the author a number of years ago. We would like to thank P. Baum for numerous conversations early on which were helpful in the development of this approach. Also, we would like to thank D. Burghelea for his continuous support.

1 Simplicial C^*-algebras

We begin by recalling some basic definitions. The simplicial category Δ is the category whose set of objects are the totally ordered sets

$$\{\underline{n} = (0 < 1 < 2 < \cdots < n)\}_{n\geq 0}$$

and whose morphisms are set maps which preserve the ordering. The augmented simplicial category Δ_+ is formed from Δ by adjoining an initial object labeled $\underline{-1}$. A simplicial object in a category $\underline{C}$ is a contravariant functor $F : \Delta \to \underline{C}$, or, equivalently, a covariant functor $\Delta^{op} \to \underline{C}$, where Δ^{op} denotes the opposite category of Δ. Similary, an augmented simplicial object in $\underline{C}$ is a covariant functor $F : \Delta_+^{op} \to \underline{C}$. A simplicial object is typically represented as a sequence of objects $\{F(\underline{n})\}_{n\geq 0}$ together with face maps $\partial_i : F(\underline{n}) \to F(\underline{n-1})$, $0 \leq i \leq n$ and degeneracy maps $s_j : F(\underline{n}) \to F(\underline{n+1})$, $0 \leq i \leq n$ satisfying the simplicial identities, while an augmented simplicial object includes an additional object $F(\underline{-1})$ together with an augmentation map $\varepsilon : F(\underline{0}) \to F(\underline{-1})$ equalizing ∂_0 and ∂_1.

Suppose $G.$ is a simplicial group. For $n \geq 1$ let $G_n^k = \bigcap_{i=0}^k ker(\partial_i : G_n \to G_{n-1})$. Let $G_0^0 = \partial_1(G_1^0)$. For notational convenience, we also set $G_n^{-1} = G_n$. Then for each $-1 \leq k < n$ there is a split-exact sequence

$$G_{n+1}^{k+1} \rightarrowtail G_{n+1}^k \twoheadrightarrow G_n^k \tag{1.1}$$

with $G_{n+1}^k \twoheadrightarrow G_n^k$ induced by ∂_{k+1} and splitting $G_n^k \rightarrowtail G_{n+1}^k$ induced by s_{k+1}. When $k = n$ there are closed sequences

$$\begin{gathered} G_{n+1}^{n+1} \rightarrowtail G_{n+1}^n \xrightarrow{\partial_{n+1}} G_n^n \\ G_0^0 \rightarrowtail G_0^{-1} = G_0 \twoheadrightarrow G_0/G_0^0 \end{gathered} \tag{1.2}$$

The (combinatorial) homotopy groups of $G.$ are defined as

$$\pi_n(G.) = G_n^n/(\partial_{n+1}(G_{n+1}^n)) \qquad n \geq 1$$

$$\pi_0(G.) = G_0/G_0^0$$

These homotopy groups agree with the usual ones, in the sense that there is a natural isomorphism of graded groups $\pi_*(G.) \cong \pi_*(|G.|, *)$ where the groups on the right are the homotopy groups of the base-pointed topological space $|G.|$ = the geometric realization of $G.$, with basepoint corresponding to $1 \in G_0$. We call $G.$ a resolution if $\pi_n(G.) = 0$ for $n > 0$. Similarly, if $(G.)_+$ is an augmented simplicial group, it is a resolution if $G.$ is one, and $\pi_0(G.) = G_{-1}$.

Throughout this paper, a $C^* - algebra$ A will always mean a normed, involutive Banach algebra over $\mathbb{C}$ satisfying $\|x\|^2 = \|x^*x\|$. C^*-algebras need not be unital. We denote by $(C^* - algebras)$ the category of C^*-algebras and C^*-algebra homomorphisms. A simplicial C^*-algebra is then a covariant functor $F : \Delta^{op} \to (C^* - algebras)$, while an augmented simplicial C^*-algebra is a covariant functor $F : \Delta_+^{op} \to (C^* - algebras)$. When referring to the homotopy groups of a simplicial C^*-algebra $A.$, we will always mean its combinatorial homotopy groups as defined above. Degreewise, this means ignoring the topology and algebra structure on A_n, treating it as a discrete abelian group under addition.

If A is a C^*-algebra, then $A^2 = A$ (cf. [Di, (1.5.8)]). In particular, if I is a closed C^*-ideal in a C^*-algebra A, it is a sub-C^*-algebra of A, so $I^2 = I$.

Proposition 1.3. *Every simplicial C^*-algebra $A.$ is a resolution.*

Proof. For $n \geq 1$, A_n^n is a closed C^*-ideal in A_n. However, for any simplicial algebra $B.$, $(B_n^n)^2 \subset \partial_{n+1}(B_{n+1}^n)$ for all $n \geq 1$. To see this, suppose $a, b \in B_n^n$. Then $s_n(a)(s_n(b) - s_{n-1}(b)) \in B_{n+1}^n$ and maps to ab under ∂_{n+1}. Consequently $\pi_n(A.) = 0$ for all $n \geq 1$. //

Note also that as A_0^0 is the image of the closed C^*-ideal A_1^0 under ∂_1, it is an involutive ideal in A_0 (by a simplicial argument), and closed (as it is the image of a C^*-homomorphism). Hence the quotient $\pi_0(A.)$ is a quotient C^*-algebra of A_0. Thus if $A.$ is a simplicial C^*-algebra, we may define its associated augmented simplicial C^*-algebra $A.^+$ to be $A.$ in non-negative degrees with $A_{-1}^+ = \pi_0(A.)$. By the above proposition, any associated augmented simplicial C^*-algebra is a resolution.

By (1.3) we have short-exact sequences of C^*-algebras and ideals:

$$A_0^0 \rightarrowtail A_0 \twoheadrightarrow \pi_0(A.) \tag{1.4}$$
$$A_n^n \rightarrowtail A_n^{n-1} \twoheadrightarrow A_{n-1}^{n-1}$$

For a Banach algebra B over $\mathbb{C}$, let $\underline{\underline{K}}(B)$ denote the topological K-theory spectrum of B [K]. We may define this as the Ω spectrum which in even dimensions is $K_0(B) \times BGL(B)$ and in odd dimensions $GL(B)$. Then $K_*^t(B) = \pi_*(\underline{\underline{K}}(B))$. The functor $B \mapsto \underline{\underline{K}}(B)$ satisfies excision, in the sense that it associates to any short-exact sequence of Banach algebras $I \rightarrowtail B \twoheadrightarrow \overline{B}$ a homotopy-fibration sequence of spectra $\underline{\underline{K}}(I) \to \underline{\underline{K}}(B) \to \underline{\underline{K}}(\overline{B})$ (loc. cit.). Thus, given a simplicial C^*-algebra $A.$, we may define a filtration on $K_*^t(\pi_0(A.))$ by

(1.5)
$$\mathcal{F}_n K_*^t(\pi_0(A.)) = ker(\partial^{(n)} : K_*^t(\pi_0(A.)) \to K_{*-n}^t((A.)_{n-1}^{n-1})) =$$
$$ker(K_*^t(\pi_0(A.)) \xrightarrow{\partial} K_{*-1}^t((A.)_0^0) \xrightarrow{\partial} K_{*-2}^t((A.)_1^1) \xrightarrow{\partial} \dots \xrightarrow{\partial} K_{*-n}^t((A.)_{n-1}^{n-1}))$$

Recall that for a discrete group π, $C^*(\pi) = C^*_{max}(\pi)$ is defined as the maximal C^*-completion of $\ell^1(\pi)$. The association $\pi \mapsto C^*(\pi)$ defines a functor from $(groups)$ (the category of discrete groups) to $(C^* - algebras)$. Thus if $\Gamma.$ is a simplicial group, applying $C^*(\_)$ degreewise produces the simplicial C^*-algebra $C^*(\Gamma.) = \{C^*(\Gamma_n)\}_{n\geq 0}$. Given $\Gamma.$, let $\pi = \pi_0(\Gamma.)$. Also associated to $\Gamma.$ is the simplicial group algebra $\mathbb{C}[\Gamma.]$. We will need the following result, due to Baum and Connes.

Lemma 1.6. *(Baum-Connes) Let $\phi : G_1 \twoheadrightarrow G_2$ be a surjective homomorphism of discrete groups. Let $I = \ker(\mathbb{C}[G_1] \overset{\phi}{\twoheadrightarrow} \mathbb{C}[G_2])$, and let I^* be the norm closure of I in $C^*(G_1)$ under the natural embedding $\mathbb{C}[G_1] \rightarrowtail C^*(G_1)$. Then there exists a canonical isomorphism of C^* algebras $A = C^*(G_1)/I^* \overset{\cong}{\to} C^*(G_2)$.*

Proof. First, A is a C^*-algebra, as it is the quotient of a C^*-algebra by a norm-closed and star-closed ideal. Because the image of any C^*-algebra homomorphism is closed [Di], it is surjective if and only if it has dense image. Thus the C^*-algebra homomorphism $C^*(\phi) : C^*(G_1) \to C^*(G_2)$ induced by ϕ is surjective, as the image contains the dense subalgebra $\mathbb{C}[G_2]$. Also, $C^*(\phi)$ sends I^* to 0, hence factors by A. Now $(I^* \cap \mathbb{C}[G_1]) \subseteq \ker(\mathbb{C}[G_1] \twoheadrightarrow \mathbb{C}[G_2] \rightarrowtail C^*(G_1)) = I$, so the inclusion

$\mathbb{C}[G_1] \rightarrowtail C^*(G_1)$ induces an inclusion $\mathbb{C}[G_2] = \mathbb{C}[G_1]/(I^* \cap \mathbb{C}[G_1]) \rightarrowtail A$ which has dense image in the norm topology. Let $\rho : G_2 \to \mathcal{B}(\mathcal{H})$ be a unitary representation of G_2 on a complex Hilbert space $\mathcal{H}$. Then $\phi \circ \rho : G_1 \to \mathcal{B}(\mathcal{H})$ is a unitary representation of G_1 which by the universal property of the maximal C^*-algebra functor $C^*(\_)$ admits a unique extension to a $*$-representation $C^*(G_1) \to \mathcal{B}(\mathcal{H})$; this representation sends I^* to 0, inducing a representation $A \to \mathcal{B}(\mathcal{H})$. Therefore A satisfies the axioms which uniquely characterize $C^*(G_2)$, implying that the C^*-algebra surjection $A \twoheadrightarrow C^*(G_2)$ is an isomorphism. //

Applying this lemma in the case $G_1 = \Gamma_0$ and $G_2 = \pi$ we conclude

Corollary 1.7. *The C^*-ideal $C^*(\Gamma.)^0_0 \overset{def}{=} \partial_1(C^*(\Gamma.)^0_1)$ is the norm-closure of $\mathbb{C}[\Gamma.]^0_0 = ker(\mathbb{C}[\Gamma_0] \twoheadrightarrow \mathbb{C}[\pi])$ in $C^*(\Gamma_0)$. Consequently, $C^*(\Gamma.)$ is a resolution of $\pi_0(C^*(\Gamma.)) = C^*(\pi)$ for any simplicial group $\Gamma.$ with $\pi = \pi_0(\Gamma.)$.*

Proof. By the previous Lemma, $C^*(\Gamma.)^0_1$ is the norm-closure of $\mathbb{C}[\Gamma.]^0_1$ in $C^*(\Gamma_1)$. Because the image of $C^*(\Gamma.)^0_1$ under ∂_1 is closed in $C^*(\Gamma_0)$, $C^*(\Gamma.)^0_0$ identifies with the norm-closure of $\partial_1(\mathbb{C}[\Gamma.]^0_1) = \mathbb{C}[\Gamma.]^0_0$ in $C^*(\Gamma_0)$. //

2 Filtering homotopy groups

As noted above, an augmented simplicial object in a category $\underline{C}$ is a covariant functor $F : \Delta^{op}_+ \to \underline{C}$. Let $n \geq -1, -1 \leq j \leq n$.

Definition 2.1. $\underline{D}^j_n \subset \Delta^{op}_+$ is the subcategory with objects $\{\underline{n}, \underline{n-1}, \dots, \underline{n-j-1}\}$. The morphisms of $\underline{D}^j_n$ are generated by

$$\{\partial_i : \underline{n-k} \to \underline{n-k-1} \mid 0 \leq i \leq j-k\} \text{ for } 0 \leq k \leq j, \text{ together with } Id_{\underline{n-k}}\,.$$

Given a functor $F : \Delta^{op}_+ \to \underline{C}$, it restricts to define a functor $F : \underline{D}^j_n \to \underline{C}$. We will need the degeneracy morphisms of Δ^{op}_+ later on in verifying certain properties of this restriction, however they are not needed for the following construction.

For an integer $m \geq 0$, $C(\underline{m})$ will denote the category of subsets of $\{0, 1, 2, \ldots, m\}$, with morphisms given by inclusions. An $\underline{m+1-\text{ cube }}$ in the category $\underline{C}$ is by definition a covariant functor $F' : C(\underline{m}) \to \underline{C}$.

Let $F : \Delta_+^{op} \to \underline{C}$ as before.

Lemma 2.2. *$F|_{\underline{D}_n^j} : \underline{D}_n^j \to \underline{C}$ determines a $(j+1)-$cube in $\underline{C}$:*

$$F_n^j : C(\underline{j}) \to \underline{C} ;$$

F_n^j is defined on morphisms and objects by

$$F_n^j(S) = F(\underline{n - |S|}),\ \phi \subseteq S \subseteq \underline{j} \ , \tag{2.3}$$

$$F_n^j(S \hookrightarrow S \cup \{i\}) = F(\partial_k) : F(\underline{n - |S|}) \to F(\underline{n - |S| - 1}) \ ,$$

where $i \notin S$ $(0 \leq i \leq j)$, and $k = i - |S_i|$, with $S_i = \{x \in S | x < i\}$.

Proof. We need to show that all possible squares commute. Let $S \subseteq \underline{j}$ be fixed, and assume $|S| \leq j - 1$. Choose i_1, i_2 with $0 \leq i_1,\ i_2 \leq j,\ i_1 \neq i_2$, and $S \cap \{i_1, i_2\} = \phi$. Now consider the square

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, tail] & S \cup \{i_1\} \arrow[d, tail] \\ S \cup \{i_2\} \arrow[r, tail] & S \cup \{i_1, i_2\} \end{tikzcd} \tag{2.4}$$

Applying F_n^j we get a square

$$\begin{tikzcd} F(\underline{n - |S|}) \arrow[r, "F(\partial_{k_1})"] \arrow[d, "F(\partial_{k_2})"'] & F(\underline{n - |S \cup \{i_1\}|}) \arrow[d, "F(\partial_{k_3})"] \\ F(\underline{n - |S \cup \{i_2\}|}) \arrow[r, "F(\partial_{k_4})"] & F(\underline{n - |S \cup \{i_1, i_2\}|}) \end{tikzcd} \tag{2.5}$$

where the subscripts k_i are determined by (2.3). We can assume without loss of generality that $i_1 < i_2$. Then

$$k_1 = i_1 - |S_{i_1}| = i_1 - |\{x \in S | x < i_1\}|$$

$$k_2 = i_2 - |S_{i_2}| = i_2 - |\{x \in S | x < i_2\}|$$

$$k_3 = i_2 - |(S \cup \{i_1\})_{i_2}| = i_2 - |\{x \in S \cup \{i_1\} | x < i_2\}|$$

$$k_4 = i_1 - |(S \cup \{i_2\})_{i_1}| = i_1 - |\{x \in S \cup \{i_2\} | x < i_1\}| \ .$$

The inequality $i_1 < i_2$ implies that $k_1 = k_4$ and $k_3 = k_2 - 1$. It also implies $k_3 \geq k_1$. The identity $\partial_{k_3}\partial_{k_1} = \partial_{k_4}\partial_{k_2}$ verifies that the square commutes. //

In what follows, $\underline{C} =$ (spaces)$_*$, (spectra)$_*$ or the category (complexes)$_*$ of chain complexes over some fixed field. We define

(2.6) i) $hpF_n^j =$ the iterated homotopy cofibre of the $(j+1)-$ cube $F_n^j : C(\underline{j}) \to \underline{C}$

ii) $hfF_n^j =$ the iterated homotopy fibre of the $(j+1)-$ cube F_n^j.

When the target is the category (spectra)$_*$ or (complexes)$_*$, the natural equivalence between homotopy-fibration sequences and homotopy-cofibration sequences produces an equivalence

$$hfF_n^j \xrightarrow{\simeq} \Omega^{j+1}\, hp\, F_n^j \ . \tag{2.7}$$

Lemma 2.8. *Let $F : \Delta_+^{op} \to \underline{C}$. Then for all $n \geq 0$ and $-1 \leq j \leq n-1$, there is a homotopy-fibration sequence*

$$hfF_n^{j+1} \xrightarrow{\overline{\alpha}_n^j} hfF_n^j \xrightarrow{\alpha_n^j} (hfF_{n-1}^j) \tag{2.9}$$

where $\overline{\alpha}_n^j$ is defined below; moreover there is an equivalence of maps

$$\alpha_n^j \simeq \widetilde{\partial}_{j+1} : hfF_n^j \to hfF_{n-1}^j \ , \tag{2.10}$$

where $\widetilde{\partial}_{j+1}$ is the map of iterated homotopy fibres induced by ∂_{j+1}.

Proof. We begin by defining inclusions of categories

(2.11) i) $C(\underline{j}) \xrightarrow{\tau_1} C(\underline{j+1})$;

$\tau_1(S) = S$, $\tau_1(S \hookrightarrow T) = (S \hookrightarrow T)$.

ii) $C(\underline{j}) \xrightarrow{\tau_2} C(\underline{j+1})$;

$\tau_2(S) = S \cup \{j+1\}$, $\tau_2(S \hookrightarrow T) = (S \cup \{j+1\} \hookrightarrow T \cup \{j+1\})$.

With F as above, and $j < n$, define

$$(F_i)_n^j : C(\underline{j}) \to \underline{C} \tag{2.12}$$

as the composition

$$C(\underline{j}) \xrightarrow{\tau_i} C(\underline{j+1}) \xrightarrow{F_n^{j+1}} \underline{C} \ .$$

It is easy to see from (2.3) that $(F_1)^j_n = F^j_n$, $(F_2)^j_n = F^j_{n-1}$. We may view the inclusion τ_2 as a natural transformation from the $(j+1)-$cube $(F_1)^j_n$ to the $(j+1)-$cube $(F_2)^j_n$; doing so yields a homotopy-fibration sequence

$$hfF^{j+1}_n \longrightarrow hf(F_1)^j_n \longrightarrow hf(F_2)^j_n \tag{2.13}$$

which, upon replacing $(F_1)^j_n$ by F^j_n and $(F_2)^j_n$ by F^j_{n-1}, agrees with the sequence in (2.9) and defines the maps α^j_n , $\overline{\alpha}^j_n$. It remains to identify α^j_n with $\widetilde{\partial}_{j+1}$ up to homotopy. As we are working up to homotopy, we may assume without loss of generality that the $(j+2)$-cube has been replaced in a functorial way by a $(j+2)$-cube in which all of the morphisms are mapped to fibrations (and the homotopy fibres are fibres). In this case, there are natural inclusions $hf(F_1)^j_n \hookrightarrow F(\phi)$, $hf(F_2)^j_n \hookrightarrow F(\{j+1\})$. By (2.3)

$$F(\phi \hookrightarrow \{j+1\}) = F(\partial_{j+1}) : F(\underline{n}) \to F(\underline{n-1}) \ , \tag{2.14}$$

which identifies α^j_n up to homotopy as the map induced by ∂_{j+1}. //

<u>Remark 2.15</u>. When $\underline{C} =$ (spaces)$_*$, care should be taken in interpreting the statement of the Lemma 2.8. In our applications of this lemma, $F(\underline{n})$ is a connected space for each $n \geq -1$. This implies hfF^j_n is connected for $j < n$. For the homotopy fibration sequence $hfF^{j+1}_n \longrightarrow hfF^j_n \longrightarrow hfF^j_{n-1}$ of (1.2.9) yields a long-exact sequence in homotopy which terminates with $\pi_1(hfF^j_{n-1})$ when $j < n-1$. However for $j = n-1,\ n > 0$, the base space may not be path-connected. In this case, the sequence should be written as

$$hfF^n_n \longrightarrow hfF^{n-1}_n \longrightarrow (hfF^{n-1}_{n-1})_0 \ .$$

If $\underline{C} =$ (spectra)$_*$ or (complexes)$_*$, this problem does not arise. Moreover in this case the natural equivalence between homotopy-fibration and homotopy-cofibration sequences in this category yield a dual formulation of (2.9) as a cofibration sequence

$$hpF^j_n \longrightarrow hpF^j_{n-1} \longrightarrow hpF^{j+1}_n \tag{2.9*}$$

where the first map is induced by ∂_{j+1}, as before.

Now from the identification $F(\underline{-1}) \simeq hfF^{-1}_{-1}$, we may use the sequences of (2.9) to define a filtration on $\pi_*(F(\underline{-1}))$.

Definition 2.16. Let $F : \Delta_+^{op} \to \underline{C}$. For $k \geq 1$ set

$$\mathcal{F}_k P_*(F(\underline{-1})) = \ker\left(\partial_*^{(k)} : P_*(hf\,F_{-1}^{-1}) \xrightarrow{\partial} P_{*-1}(hf\,F_0^0) \xrightarrow{\partial} \dots \xrightarrow{\partial} P_{*-k}(hf\,F_{k-1}^{k-1})\right)$$

where $P_*(\_) = \pi_*(\_)$ for $\underline{C} = (\text{spaces})_*$ or $(\text{spectra})_*$, and $H_*(\_)$ for $\underline{C} = (\text{complexes})_*$. The maps which appear in the sequence are the boundary maps associated with the sequence (2.9) for $-1 \leq j = n-1 \leq k-1$.

This is clearly an increasing filtration of $\pi_*(F(\underline{-1}))$. When $\underline{C}$ is the category of spaces, $\pi_*(F(\underline{-1})) = \varinjlim_k \mathcal{F}_k \pi_*(F(\underline{-1}))$ for dimensional reasons. When $\underline{C} = (\text{spectra})_*$ or $(\text{complexes})_*$, the filtration of $\pi_*(F(\underline{-1}))$ is more natural for the reason that long-exact sequences are allowed to continue on indefinitely, preserving exactness. Because of the functoriality of the constructions involved, we have

Proposition 2.17. *A natural transformation $F_1 \xrightarrow{\zeta} F_2 : \Delta_+^{op} \to \underline{C}$ induces compatible natural transformations $(F_1)_n^j \to (F_2)_n^j$, and hence a filtration-preserving homomorphism of homotopy groups*

$$\mathcal{F}_*\zeta_* : \mathcal{F}_*\pi_*(F_1(\underline{-1})) \longrightarrow \mathcal{F}_*\pi_*(F_2(\underline{-1}))\ .$$

Proof. Clear. //

Finally, we note the following fact, which will be used often in what follows.

Proposition 2.18. *If $F : \Delta_+^{op} \to \underline{C} = (\text{complexes})_*$, and for each k the simplicial abelian group $\{[n] \mapsto ((F(n))_k\}_{n\geq 0}$ is a resolution (i.e., has vanishing homotopy groups above dimension 0), then for each $-1 \leq j \leq n$ there is a weak equivalence*

$$fF_n^j \overset{\simeq}{\rightarrowtail} hfF_n^j$$

where fF_n^j denotes the iterated fibre of the $(j+1)$- cube F_n^j, and the map is the natural inclusion of the iterated fibre into the iterated homotopy fibre.

Proof. This follows by induction on j starting with $j = -1$ and showing that for each k

$$fG_n^{j+1} \xrightarrow{\overline{\alpha}_n^j} fG_n^j \xrightarrow{\alpha_n^j} fG_{n-1}^j$$

is a fibration sequence, where $G(n) = F(n)_k$. For $j < n$ this follows immediately from the simplicial identities which show that the map on the right is a surjection of simplicial abelian groups. In the final case $j = n$, the surjectivity of this map follows from the hypothesis on F. //

3 A basic example

We will identify the filtration of the preceding section in a simple but important case.

For a simplicial set $X.$, let $\mathbb{Z}\{X.\}$ denote the free simplicial abelian group generated by $X.$. The chain complex associated to $\mathbb{Z}\{X.\}$ is exactly the singular chain complex for $X.$, so $\pi_*(\mathbb{Z}\{X.\}) \cong H_*(X.)$. For a discrete group G, BG will denote the standard simplicial (non-homogeneous) bar construction on G.

In this section $\Gamma.$ denotes a free simplicial resolution of a discrete group π. Define $G_n^j B(\Gamma.)$ for $n \geq 1$ by

$$G_n^j B(\Gamma.) = \bigcap_{k=0}^{j} \ker\left(\partial_k : \mathbb{Z}\{B\Gamma_n\} \to \mathbb{Z}\{B\Gamma_{n-1}\}\right) . \tag{3.1}$$

Thus $G_n^j B(\Gamma.)$ is a simplicial abelian subgroup of $\mathbb{Z}\{B\Gamma_n\}$ for $0 \leq j \leq n$, $n \geq 1$. Similarly we define

$$\begin{aligned} G_0^0 B(\Gamma.) &= \ker(\mathbb{Z}\{B\Gamma_0\} \twoheadrightarrow \mathbb{Z}\{B\pi\}) \ , \\ G_n^{-1} B(\Gamma.) &= \mathbb{Z}\{B\Gamma_n\} \ \text{ for } n \geq 0 \ , \\ G_{-1}^{-1} B(\Gamma.) &= \mathbb{Z}\{B\pi\} \ . \end{aligned} \tag{3.2}$$

<u>Lemma 3.3</u>. *i) For each $n \geq 0$ and $-1 \leq j \leq n-1$, there is a short-exact sequence of simplicial abelian groups*

$$G_n^{j+1} B(\Gamma.) \rightarrowtail G_n^j B(\Gamma.) \overset{(\partial_{j+1})_*}{\twoheadrightarrow} G_{n-1}^j B(\Gamma.) \tag{3.4}$$

ii) For $0 \leq j \leq n$, $\pi_0(G_n^j B(\Gamma.)) = 0$. For $0 \leq j \leq n-1$, $\pi_i(G_n^j B(\Gamma.)) = 0$ when $i > 1$.

Proof. Certainly $G_n^{j+1}B(\Gamma.)$ is the kernel of $(\partial_{j+1})_*$ restricted to $G_n^jB(\Gamma.)$. The simplicial identities imply that $(\partial_i)_* \circ (\partial_{j+1})_* = (\partial_j)_* \circ (\partial_i)_*$ for $i \leq j$, so the image of $(\partial_{j+1})_* \,|_{G_n^jB(\Gamma.)}$ is contained in $G_{n-1}^jB(\Gamma.)$.

If $j < n-1$ and $x \in G_{n-1}^jB(\Gamma.)$, then $(s_{j+1})_*(x) \in G_n^jB(\Gamma)$ and maps to x under $(\partial_{j+1})_*$, proving surjectivity in this case.

Now fix p, and consider the simplicial abelian group $\{\mathbb{Z}\{(B\Gamma_n)_p\}\}_{n\geq 0}$. $(B\Gamma_n)_p \cong (\Gamma_n)^p$, and the simplicial set $\{(B\Gamma_n)_p\}_{n\geq 0}$ is isomorphic as a simplicial set to the simplicial group $\{(\Gamma_n)^p\}_{n\geq 0} \cong (\Gamma.)^p$ equipped with diagonal simplicial structure. Thus the simplicial abelian group $\{\mathbb{Z}\{(B\Gamma_n)_p\}\}_{n\geq 0}$ identifies with the underlying abelian group of the simplicial group algebra $\mathbb{Z}[(\Gamma.)^p]$. As $\Gamma. \twoheadrightarrow \pi$ is a weak equivalence, $(\Gamma.)^p \xrightarrow{\cong} \pi^p$ is again a weak equivalence for each $p > 0$. Thus $\pi_i(\mathbb{Z}[(\Gamma.)^p]) = 0$ for all $i > 0$ and $p > 0$. The first part of the lemma now follows by the same argument as in Proposition 2.18.

For each $n \geq 0$, $B\Gamma_n$ is a 0-reduced simplicial set. Thus $\mathbb{Z}\{B\Gamma_n\}_0 \cong \mathbb{Z}$ for all $n \geq 0$; moreover the simplicial structure on $\{\mathbb{Z}\{B\Gamma_n\}_0\}_{n\geq 0}$ induced by the face and degeneracy maps of $\Gamma.$ is the trivial one. It follows that $G_n^jB(\Gamma.)_0 = 0$ for all $n,\ 0 \leq j \leq n$ so $\pi_0(G_n^jB(\Gamma.)) = 0$. Now suppose $n \geq 1$.

The sequence

$$G_n^0B(\Gamma.) \longrightarrow \mathbb{Z}\{B\Gamma_n\} \overset{(\partial_0)_*}{\twoheadrightarrow} \mathbb{Z}\{B\Gamma_{n-1}\} \tag{3.5}$$

is split-exact ($(s_0)_*$ splits $(\partial_0)_*$). Γ_j is a free group for each j so $\pi_i(\mathbb{Z}\{B\Gamma_j\}) = H_i(B\Gamma_j) = 0$ for $i > 1,\ j \geq 0$. The split-exactness of (3.5) implies $\pi_i(G_n^0B(\Gamma.)) = 0$ for $i > 1,\ n \geq 0$. Inductively, assume the result for $j-1$. By the above there is a short exact sequence

$$G_n^jB(\Gamma.) \rightarrowtail G_n^{j-1}B(\Gamma.) \overset{(\partial_j)_*}{\twoheadrightarrow} G_{n-1}^{j-1}B(\Gamma.) \ . \tag{3.6}$$

which is split-exact for $j < n$. By the five lemma, $\pi_i(G_n^jB(\Gamma.)) = 0$ for $i > 1$. //

We define functors $F_i,\ i = 0, 1, 2$ by

(3.7)

(1) $F_0 : \Delta_+^{op} \to (groups)$, $F_0(\underline{n}) = \Gamma_n \;\; n \geq 0$, $F_0(\underline{-1}) = \pi$. Thus $F_0|_{\Delta^{op}}$ is the simplicial group $\Gamma.$ and $F_0(\underline{0} \to \underline{-1}) = \epsilon : \Gamma_0 \twoheadrightarrow \pi$.

(2) $F_1 = \mathbb{Z}\{BF_0\}$. Thus for each $n \geq -1$, $F_1(\underline{n})$ is the simplicial abelian group $\mathbb{Z}\{BF_0(\underline{n})\}$.

(3) F_2 is the augmented simplicial functor which assigns to n the associated chain complex of $F_1(\underline{n})$, with face and degeneracy maps induced by $\Gamma.$.

By the previous section, the restriction of F_i to the subcategory $\underline{D}_n^j \subset \Delta_+^{\mathrm{op}}$ determines a $(j+1)-$cube $(F_i)_n^j : C(\underline{j}) \to \underline{C}$, where $\underline{C}$ is the target of F_i.

<u>Lemma 3.8</u>. *For all $n \geq -1$, $-1 \leq j \leq n$, there are weak equivalences*

$$hf(F_1)_n^j \xleftarrow{\simeq} f(F_1)_n^j = G_n^j B(\Gamma.)$$
$$hf(F_2)_n^j \xleftarrow{\simeq} f(F_2)_n^j = (G_n^j B(\Gamma.))_*$$

where $(A.)_*$ denotes the associated chain complex of the simplicial abelian group $A.$. Under this equivalence, the homotopy fibration sequence of (2.9) is identified with the fibration sequence of (3.4), up to homotopy.

<u>Proof</u>. By definition, $hf(F_1)_n^{-1} \xrightarrow{=} G_n^{-1}B(\Gamma.)$. By Proposition 2.18, the hypothesis on $\Gamma.$ implies that the natural map from the iterated fibre – $(G_n^j B(\Gamma.))_*$ – to the iterated homotopy fibre $hf(F_1)_n^j$ is a weak equivalence for all $j \leq n$. The same argument works for F_1. //

The construction in Definition 2.16 produces a filtration on $\pi_*(F_1(\underline{-1})) = H_*(B\pi)$ which may alternatively be defined using the fibration sequences in (3.4). Lemma 3.3 and the fact that $\pi_i(G_n^j B(\Gamma.)) = 0$ for $i > 0$ and $j < n$ implies that $\partial : \pi_i(G_{k-1}^{k-1}B(\Gamma.)) \to \pi_{i-1}(G_k^k B(\Gamma.))$ is an isomorphism if $i > 2$ and injective if $i = 2$. Thus $\pi_j(\mathbb{Z}\{B\pi\})$ maps injectively by $\partial^{(j-1)}$ to $\pi_1(G_{j-2}^{j-2}B(\Gamma.))$, and then to zero under $\partial : \pi_1(G_{j-2}^{j-2}B(\Gamma.)) \to \pi_0(G_{j-1}^{j-1}B(\Gamma.)) = 0$. It follows that $0 \neq x \in \pi_j(\mathbb{Z}\{B\pi\})$ maps to zero under $\partial^{(n)}$ if and only if $j \leq n$. The same remarks apply to the functor F_2. We conclude

Proposition 3.9. *The filtration on*

$$H_*(F_2(\underline{-1})) = \pi_*(F_1(\underline{-1})) = H_*(B\pi;\mathbb{Z})$$

defined by (2.16) agrees with that induced by the skeletal filtration of $B\pi$:

$$\mathcal{F}_n H_*(F_2(\underline{-1})) = \mathcal{F}_n \pi_*(F_1(\underline{-1})) = \mathcal{F}_n H_*(B\pi;\mathbb{Z}) = \bigoplus_{i=0}^{n} H_i(B\pi;\mathbb{Z}) \ .$$

4 Filtering the assembly map

As above, let $\Gamma.^+$ be an augmented free simplicial resolution of π. Define functors $F_i : \Delta_+^{op} \to (spectra)_*$ by $F_1(\underline{n}) = \underline{\underline{KU}}(B\Gamma_n)$ = the (unreduced) 2-periodic complex K-homology spectrum of the classifying space $B\Gamma_n$, and $F_2(\underline{n}) = \underline{\underline{K}}^t(C^*(\Gamma_n))$ = the 2-periodic topological K-theory spectrum of $C^*(\Gamma_n)$. The assembly map determines a natural transformation

$$\mathcal{A} : F_1 \to F_2 \tag{4.1}$$

By (2.16), $\Gamma.^+$ determines filtrations on both $\pi_*(F_1(\underline{-1})) = KU_*(B\pi)$ and $\pi_*(F_2(\underline{-1})) = K_*^t(C^*(\pi))$, and by Proposition 2.17, $\mathcal{A}_*(\pi)$ = the map on homotopy groups induced by $\mathcal{A}(\pi)$, is filtration-preserving.

By (1.5) and (1.7), the simplicial C^*-algebra $C^*(\Gamma.^+)$ also induces a filtration on $K_*^t(C^*(\pi))$. Our first observation is that that this filtration agrees with that given by (2.16). Precisely, it follows from excision in topological K-theory that for each $-1 \le k \le n$ there is a commuting diagram

$$\begin{tikzcd}
\underline{\underline{K}}^t(C^*(\Gamma.^+)_n^k) \arrow[r] \arrow[d, "\simeq"] & \underline{\underline{K}}^t(C^*(\Gamma.^+)_n^{k-1}) \arrow[r, "\widetilde{\partial_k}"] \arrow[d, "\simeq"] & \underline{\underline{K}}^t(C^*(\Gamma.^+)_{n-1}^{k-1}) \arrow[d, "\simeq"] \\
hf(F_2)_n^k \arrow[r] & hf(F_2)_n^{k-1} \arrow[r, "\widetilde{\partial_k}"] & hf(F_2)_{n-1}^{k-1}
\end{tikzcd}$$

where each row is a homotopy-fibration sequence and each vertical arrow is a weak equivalence. This implies the equality of the sequence in (1.5) with that in (2.16). The assembly map induces maps of homotopy-fibers

$$hf\mathcal{A}_n^k : hf(F_1)_n^k \to hf(F_2)_n^k \tag{4.2}$$

For $k = -1$ and $n \geq 0$ this is the assembly map for a free group, which is a weak equivalence. The homotopy-fibration sequence of (2.9) is naturally split up to homotopy by the degeneracy s_{j+1}. Thus $hf\mathcal{A}_n^k$ is a weak equivalence for all $-1 \leq k < n$. Together with an easy diagram-chase, this implies that if $x \in KU_*(B\pi)$ and $\mathcal{A}_*(\pi)(x) \in \mathcal{F}_n K_*^t(C^*(\pi))$, then $x \in \mathcal{F}_n KU_*(B\pi)$ (in fact this follows simply from the surjectivity of $\mathcal{A}_*(F)$ for free groups F). Finally, via the functorial Atiyah-Hirzebruch Chern isomorphism $KU_*(B\pi) \otimes \mathbb{Q} \underset{\simeq}{\xrightarrow{ch}} \bigoplus_{i\in\mathbb{Z}} H_{*-2i}(B\pi;\mathbb{Q})$, the computation in the previous section identifies, at least rationally, the filtration on $KU_*(B\pi)$ induced by $\Gamma.^+$ as the skeletal filtration

$$\mathcal{F}_n KU_*(B\pi) \otimes \mathbb{Q} = \bigoplus_{\substack{i\in\mathbb{Z} \\ *-2i\leq n}} H_{*-2i}(B\pi;\mathbb{Q}) \tag{4.3}$$

Summarizing, we have

Theorem 4.4. *Any free augmented simplicial resolution $\Gamma.^+$ of a discrete group π induces a filtration of both $KU_*(B\pi)$ and $K_*^t(C^*(\pi))$. The assembly map preserves this filtration. The filtration on $KU_*(B\pi)$ identifies rationally with the skeletal filtration (4.3), while the filtration on $K_*^t(C^*(\pi))$ is defined as in (1.5). Finally, if $x \in KU_*(B\pi)$ and $\mathcal{A}_*(\pi)(x) \in \mathcal{F}_n K_*^t(C^*(\pi))$, then $x \in \mathcal{F}_n KU_*(B\pi)$.*

As indicated in the introduction, the assembly map $\mathcal{A}_*(\pi)$ is rationally injective iff it is rationally injective on $ku_*(B\pi)\otimes\mathbb{Q}$, the rational connective K-homology groups of $B\pi$. As a first corollary to the previous theorem we have

Corollary 4.5. *The assembly map resticted to connective K-homology $\mathcal{A}_*(\pi)$: $ku_*(B\pi) \to K_*^t(C^*(\pi))$ is rationally injective iff the composition*

$$H_n(B\pi;\mathbb{Q}) \rightarrowtail ku_n(B\pi)\otimes\mathbb{Q} \xrightarrow{\mathcal{A}_n(\pi)_{\mathbb{Q}}} K_n^t(C^*(\pi)) \tag{4.6}$$

is injective for all $n \geq 1$, where the first map is the inclusion of the top-dimensional summand under the isomorphism $ku_n(B\pi)\otimes\mathbb{Q} \cong \oplus_{i\geq 0} H_{n-2i}(B\pi;\mathbb{Q})$.

Proof. We assume given an element of $ku_n(B\pi)\otimes\mathbb{Q}$, written as $x = (x_n, x_{n-2}, \dots, x_\varepsilon)$ where $x_n \neq 0$, $x_m \in H_m(B\pi;\mathbb{Q})$ and $\varepsilon = 0$ or 1. The image of this element under

the rationalized assembly map lies in $\mathcal{F}_n K_n^t(C^*(\pi))\otimes\mathbb{Q}$, and under the composition

$$\begin{aligned}ku_n(B\pi)\otimes\mathbb{Q}=\mathcal{F}_n KU_n(B\pi)\otimes\mathbb{Q} &\xrightarrow{\mathcal{A}_n(\pi)_{\mathbb{Q}}} \mathcal{F}_n K_n^t(C^*(\pi))\otimes\mathbb{Q}\\ &\twoheadrightarrow \big(\mathcal{F}_n K_n^t(C^*(\pi))\otimes\mathbb{Q}\big)\,/\,\big(\mathcal{F}_{n-1} K_n^t(C^*(\pi))\otimes\mathbb{Q}\big)\end{aligned}$$

has the same image as the element $\widetilde{x}=(x_n,0,0,\ldots,0)$. Suppose $\mathcal{A}_n(\pi)_{\mathbb{Q}}(\widetilde{x})\neq 0$ in $\mathcal{F}_n K_n^t(C^*(\pi))\otimes\mathbb{Q}$, but maps to zero in the quotient $(\mathcal{F}_n K_n^t(C^*(\pi))\otimes\mathbb{Q})\,/\,(\mathcal{F}_{n-1} K_n^t(C^*(\pi))\otimes\mathbb{Q})$. This would imply $\mathcal{A}_n(\pi)_{\mathbb{Q}}(\widetilde{x})\in\mathcal{F}_{n-1} K_n^t(C^*(\pi))\otimes\mathbb{Q}$, which by the previous theorem would imply $\widetilde{x}\in\mathcal{F}_{n-1} KU_n(C^*(\pi))\otimes\mathbb{Q}$ - a contradiction (again by the previous theorem). Therefore if $\widetilde{x}$ maps non-trivially to $\mathcal{F}_n K_n^t(C^*(\pi))\otimes\mathbb{Q}$, it maps non-trivially to $(\mathcal{F}_n K_n(C^*(\pi))\otimes\mathbb{Q})\,/\,(\mathcal{F}_{n-1} K_n(C^*(\pi))\otimes\mathbb{Q})$, and therefore so does x. The other direction is obvious. //

Referring to (1.5), we see that $(\mathcal{F}_n K_n^t(C^*(\pi))\otimes\mathbb{Q})\,/\,(\mathcal{F}_{n-1} K_n^t(C^*(\pi))\otimes\mathbb{Q})$ injects into $K_1^t(C^*(\Gamma_\cdot^+)_{n-2}^{n-2})\otimes\mathbb{Q}$, and then maps to zero under the boundary map to $K_0^t(C^*(\Gamma_\cdot^+)_{n-1}^{n-1})\otimes\mathbb{Q}$. In other words, it factors by the inclusion

(4.7)

$$\textit{coker}\,\big(K_1^t(C^*(\Gamma_\cdot^+)_{n-1}^{n-1})\otimes\mathbb{Q}\to K_1^t(C^*(\Gamma_\cdot^+)_{n-1}^{n-2})\otimes\mathbb{Q}\big)\rightarrowtail K_1^t(C^*(\Gamma_\cdot^+)_{n-2}^{n-2})\otimes\mathbb{Q}$$

Thus the last corollary can be further refined to

Corollary 4.8. *The composition*

$$\begin{aligned}H_n(B\pi;\mathbb{Q})\rightarrowtail \mathcal{F}_n KU_n(B\pi)\otimes\mathbb{Q} &\xrightarrow{\mathcal{A}_n(\pi)_{\mathbb{Q}}} \mathcal{F}_n K_n^t(C^*(\pi))\\ &\to \textit{coker}\,\big(K_1^t(C^*(\Gamma_\cdot^+)_{n-1}^{n-1})\otimes\mathbb{Q}\to K_1^t(C^*(\Gamma_\cdot^+)_{n-1}^{n-2})\otimes\mathbb{Q}\big)\end{aligned}\tag{4.9}$$

is injective for a given n if and only if (4.6) is injective for that n.

In terms of absolute K-groups, another corollary of the above is

Colollary 4.10. *The Strong Novikov Conjecture holds for $C^*(\pi)$ if and only if the composition*

$$H_n(B\pi;\mathbb{Q})\xrightarrow{\mathcal{A}(\pi)_{\mathbb{Q}}} K_n^t(C^*(\pi))\otimes\mathbb{Q}\xrightarrow{\partial^{(n-1)}} K_1^t(C^*(\pi)_{n-2}^{n-2})\otimes\mathbb{Q}$$

is injective for each $n\geq 1$

One possible method for detecting this image is via the map on $K^t_1(\_)$ induced by the surjection $C^*(\pi)^{n-2}_{n-2} \twoheadrightarrow C^*_{ab}(\pi)^{n-2}_{n-2}$ where $C^*_{ab}(\pi)^{n-2}_{n-2}$ denotes the C^*-algebra abelianization of $C^*(\pi)^{n-2}_{n-2}$ (i.e., the quotient by the norm-closure of the commutator ideal).

5 Simplicial C^i-algebras and the Strong Novikov Conjecture

Let G be a discrete group. We define $C^i(G)$ as the inverse-closure of $\mathbb{C}[G]$ in $C^*(G)$. In other words, i) $C^i(G)$ contains $\mathbb{C}[G]$ and ii) if $a \in C^i(G)$ and $a^{-1} \in C^*(G)$, then $a^{-1} \in C^i(G)$. This construction is clearly functorial with respect to group homomorphisms. By a [Sch], $M_n(C^i(G))$ is then inverse-closed in $M_n(C^*(G))$ for all n, where $M_n(A)$ denotes the ring of $n \times n$ matrices with coefficients in A. From the resulting equality

$$GL_n(C^i(G)) = M_n(C^i(G)) \cap GL_n(C^*(G)) \qquad n \geq 1$$

we conclude the inclusion $C^i(G) \hookrightarrow C^*(G)$ induces a weak equivalence of topological groups

$$GL(C^i(G)) \overset{\simeq}{\hookrightarrow} GL(C^*(G)) \tag{5.1}$$

and hence an isomorphism of higher topological K-groups

$$K^t_*(C^i(G)) \overset{\cong}{\to} K^t_*(C^*(G)) \qquad * \geq 1$$

(the topology on $C^i(G)$ is the fine topology, unless stated otherwise. Because $C^i(G)$ is inverse-closed in $C^*(G)$, its higher topological K-groups are the same for any topology between the fine topology and the induced norm topology [O1, App.]). The assignment $G \mapsto C^i(G)$ is functorial with respect to group homomorphisms, hence extends to simplicial and augmented simplicial groups.

As in the previous section, $\Gamma.^+$ denotes a free augmented simplicial resolution of $\Gamma_{-1} = \pi$, and $C^i(\Gamma.^+)$ the resulting augmented simplicial algebra formed by applying $C^i(\_)$ degreewise. The corresponding simplicial group $\Gamma.$ is gotten by omitting the degree -1 part.

Lemma 5.2. *There is a factorization*

$$(5.3)\qquad \begin{aligned} H_n(B\pi;\mathbb{Q}) &\xrightarrow{\mathcal{A}_n(\pi)_{\mathbb{Q}}} \mathcal{F}_n K_n^t(C^*(\pi)) \\ &\to coker\left(K_1^t(\partial_n(C^i(\Gamma.^+)_n^{n-1}))\otimes\mathbb{Q}\to K_1^t(C^i(\Gamma.^+)_{n-1}^{n-2})\otimes\mathbb{Q}\right) \\ &\xrightarrow{\cong} coker\left(K_1^t(C^*(\Gamma.^+)_{n-1}^{n-1})\otimes\mathbb{Q}\to K_1^t(C^*(\Gamma.^+)_{n-1}^{n-2})\otimes\mathbb{Q}\right) \end{aligned}$$

Proof. The inclusion $C^i(\Gamma.^+)\hookrightarrow C^*(\Gamma.^+)$ of augmented simplicial algebras yields inclusions $C^i(\Gamma.^+)_n^k\hookrightarrow C^*(\Gamma.^+)_n^k$ which induce isomorphisms in higher topological K-theory

$$K_*(C^i(\Gamma.^+)_n^k)\cong K_*(C^*(\Gamma.^+)_n^k)\qquad *\geq 1$$

The case $k<n$ follows from the case $k=-1$. When $k=n$, it is clear that $C^i(\Gamma.^+)_n^n$ is inverse-closed in $C^*(\Gamma.^+)_n^n$. By Lemma 1.6, it is dense in the norm topology. Now the inclusion $\partial_n(C^i(\Gamma.^+)_n^{n-1})\hookrightarrow C^i(\Gamma.^+)_{n-1}^{n-1}$ may not be surjective (in fact, this is an open question; see below). However, it also induces an isomorphism in higher topological K-theory. This is because $\partial_n(C^i(\Gamma.^+)_n^{n-1})\hookrightarrow\partial_n(C^*(\Gamma.^+)_n^{n-1})$ does, and $\partial_n(C^*(\Gamma.^+)_n^{n-1})=C^*(\Gamma.^+)_{n-1}^{n-1}$. The result follows from (4.7). //

In conjunction with section 1, the previous lemma implies

Theorem 5.4. *The Strong Novikov Conjecture is true for π if and only if the map in (5.3)*

$$H_n(B\pi;\mathbb{Q})\xrightarrow{\widetilde{\mathcal{A}}^t(\pi)_n} coker\left(K_1^t(\partial_n(C^i(\Gamma.^+)_n^{n-1}))\otimes\mathbb{Q}\to K_1^t(C^i(\Gamma.^+)_{n-1}^{n-2})\otimes\mathbb{Q}\right)$$

is injective for each $n\geq 1$.

We denote by $K(\mathbb{Q},n-1)$. the simplicial abelian group which is $\{id\}$ in degrees less than $n-1$, equal to $\mathbb{Q}$ in degree $n-1$, and equal to $\underset{s\in S_{n-1,m}}{\oplus}\mathbb{Q}_s$ in degree $m\geq n$, where $S_{n-1,m}$ is the set of distinct iterated degeneracies from dimension $n-1$ to m, and $\mathbb{Q}_s$ indicates the copy of $\mathbb{Q}$ gotton by applying s to the unique copy of $\mathbb{Q}$ in dimension $n-1$. There are isomorphisms

(5.5)

$$H^n(B\pi;\mathbb{Q})=[B\pi,K(\mathbb{Q},n)]_*=[|B\Gamma.|,K(\mathbb{Q},n)]_*\cong Hom_{s.gps}(\Gamma.,K(\mathbb{Q},n-1).)$$

where $[X, Y]_*$ denotes (basepointed) homotopy classes of maps between (basepointed) spaces X and Y, while the right-hand side denotes the (abelian group) of simplicial group homomorphisms from $\Gamma.$ to $K(\mathbb{Q}, n-1).$. The first two equalities are well-known, while the third follows by an easy simplicial argument. And as we have noted, $C^i(\_)$ is functorial, so a simplicial group homomorphism $\phi : \Gamma. \to K(\mathbb{Q}, n-1).$ induces a continuous homomorphism of C^i-algebras

$$(5.6) \qquad \widetilde{\phi} : C^i(\Gamma.) \to C^i(K(\mathbb{Q}, n-1).)$$

The following useful fact was communicated to the author by P. Baum [B].

Lemma 5.7. *For a discrete group G, let $I_{\mathbb{C}}[G] = ker(\varepsilon : \mathbb{C}[G] \to \mathbb{C})$, and $I^i(G) = ker(\varepsilon : C^i(G) \to \mathbb{C})$. Then the canonical inclusion $\mathbb{C}[G] \hookrightarrow C^i(G)$ induces an isomorphism*

$$(5.8) \qquad I_{\mathbb{C}}[G]/(I_{\mathbb{C}}[G])^2 \xrightarrow{\cong} I^i(G)/(I^i(G))^2$$

Proof. Because the map $G \to G_{ab} = G/[G, G]$ induces an isomorphism on the respective quotients in (5.8), it suffices to consider the case when G is abelian, and one may further restrict to the case G is torsion-free. By considering elements in $\mathbb{C}[G]$ of the form $1 + a$ where $a \in I_{\mathbb{C}}[G]$ and $|a| < 1$ ($|a|$ = the C^*-norm of a) one sees the map in (5.8) is surjective. Injectivity follows from the finitely-generated case by a direct limit argument. Finally, the finitely-generated torsion-free case reduces to the case $G = \mathbb{Z}$ which is easy to verify directly. //

Suppose given a homomorphism $\widetilde{\phi}$ of simplicial algebras as in (5.6). It is easy to see that $C^i(K(\mathbb{Q}, n-1).)^{n-1}_{n-1} = I^i(\mathbb{Q})$ and $\partial_n(C^i(K(\mathbb{Q}, n-1).)^{n-1}_n) = (I^i(\mathbb{Q}))^2$. Let $K^a_*(R)$ denote the algebraic K-groups of the discrete ring R. We then have an induced homomorphism

(5.9)
$$(\widetilde{\phi})^a_1 : coker\left(K^a_1(\partial_n(C^i(\Gamma.^+)^{n-1}_n)) \otimes \mathbb{Q} \to K^a_1(C^i(\Gamma.^+)^{n-2}_{n-1}) \otimes \mathbb{Q}\right)$$
$$\longrightarrow coker\left(K^a_1(\partial_n(C^i(K(\mathbb{Q}, n-1).)^{n-1}_n)) \otimes \mathbb{Q} \to K^a_1(C^i(K(\mathbb{Q}, n-1).)^{n-2}_{n-1}) \otimes \mathbb{Q}\right)$$
$$\longrightarrow K^a_1(I^i(\mathbb{Q})/(I^i(\mathbb{Q}))^2) \otimes \mathbb{Q} \cong \mathbb{C}$$

This composition is the analogue in algebraic K-theory which, if it could be extended to topological K-theory would imply injectivity of the composition in (1.5.4) in all dimensions. Our next task is to formulate a version of the statement in Theorem 5.4 which is true in algebraic K-theory. To do so, we first give an alternative description of $H_*(B\pi;\mathbb{Q})$. Let $\{D_n\}_{n\geq 0}$ be the chain complex $D_n = H_1(B\Gamma_{n-1}^{n-2};\mathbb{Q})$, $d_n^D = (\partial_{n-1})_* : D_n \to D_{n-1}$. For $n < 1$ set $D_n = 0$. For each $n \geq 1$, Γ_{n-1} acts on Γ_{n-1}^{n-2} by conjugation. This defines an action of Γ_{n-1} on D_n; let $\overline{D}_n = (D_n)_{\Gamma_{n-1}}$, the coinvariants with respect to the action. It is straightforward to verify that $d_n^D = (\partial_{n-1})_*$ induces a map on coinvariants, yielding a complex $\overline{D}_* = \{\overline{D}_n, d_n^{\overline{D}}\}$. Let $\{E_n\}_{n\geq 0}$ be the chain complex $E_n = H_1(B\Gamma_{n-1};\mathbb{Q})$, $d_n^E = \sum_{i=0}^{n-1}(-1)^i(\partial_i)_* : E_n \to E_{n-1}$, with $E_n = 0$ for $n < 1$. Finally, let $\overline{E}_n$ denote the quotient of E_n by the subgroup generated by elements of the form $(s_j)_*(x)$ where $x \in H_1(B\Gamma_{n-2};\mathbb{Q})$ and $(s_j)_*$ the map induced on homology by the degeneracy s_j. Again, d_n^E descends to a map $d_n^{\overline{E}} : \overline{E}_n \to \overline{E}_{n-1}$, making $\overline{E}_* = \{\overline{E}_n, d_n^{\overline{E}}\}$ a chain complex.

Lemma 5.10. *There are isomorphisms*

$$H_*(\overline{D}_*) \cong H_*(E_*) \cong H_*(\overline{E}_*) \cong H_*(B\pi;\mathbb{Q})$$

Proof. First, $\overline{E}_*$ is the normalized complex of E_* formed by collapsing the acyclic subcomplex of degenerate elements, so $H_*(E_*) \cong H_*(\overline{E}_*)$. The E^1-term of the spectral sequence in homology $\{E^1_{p,q} = H_p(B\Gamma_q;\mathbb{Q})\}$ converging to $H_*(B\Gamma.;\mathbb{Q}) \cong H_*(B\pi;\mathbb{Q})$ vanishes for $p > 1$ (as Γ_n is free for all $n \geq 0$); upon passing to the E^2-term, the $p = 0$ line vanishes while the $p = 1$ line yields precisely the homology of E_*. Thus $H_*(E_*) \cong H_*(B\pi;\mathbb{Q})$.

To see the relationship between $H_*(\overline{D})$ and $H_*(\overline{E}_*)$, we construct maps in both directions. Let $\lambda_n^j = s_j\partial_j : \Gamma_n \to \Gamma_n$ where $0 \leq j < n$. Inductively define $r_n^j(g)$ by $r_n^{-1}(g) = g$; $r_n^j(g) = (r_n^{j-1}(g))(\lambda_n^j(r_n^{j-1}(g)))^{-1}$. For each $j < n$, r_n^j defines a projection of sets $r_n^j : \Gamma_n \twoheadrightarrow \Gamma_n^j$. Let $\Gamma_n^{j,d}$ denote the normal subgroup of Γ_n^j generated by commutators of the form $[x, y]$ where $x \in \Gamma_n^j$ and $y = s_i(z)$ for some

$z \in \Gamma_{n-1}$ and $0 \leq i \leq j$. Let $\overline{\Gamma}^j_n = \Gamma^j_n / \Gamma^{j,d}_n$. An application of simplicial identities verifies that the composition $\Gamma_n \xrightarrow{r^j_n} \Gamma^j_n \twoheadrightarrow \overline{\Gamma}^j_n$ is a homomorphism. There is an equality $H_1(B\Gamma^{n-1}_n;\mathbb{Q})_{\Gamma_n} = H_1(B\overline{\Gamma}^{n-1}_n;\mathbb{Q})$, from which we see that r^{n-2}_{n-1} induces for each $n \geq 1$ a homomorphism $(r^{n-2}_{n-1})_* : E_n \to \overline{D}_n$. Again, use of simplicial identities allows one to work out the following recursive description for $\partial_n(r^{n-1}_n(g))$: let $A_0(g) = r^{n-2}_{n-1}(\partial_0(g))$, with $A_i(g) = r^{n-2}_{n-1}(\partial_i(g))A_{i-1}(g)^{-1}$. Then $\partial_n(r^{n-1}_n(g)) = A_n(g)$. This formula implies $\{(r^{n-2}_{n-1})_*\}_{n\geq 1}$ defines a chain map $E_* \to \overline{D}_*$. On the other hand, the inclusion $\Gamma^{n-1}_n \hookrightarrow \Gamma_n$ for each n defines a chain map $D_* \to E_*$ which factors by the projection $D_* \twoheadrightarrow \overline{D}_*$, as $H_1(B\Gamma_n;\mathbb{Q}) = H_1(B\Gamma_n;\mathbb{Q})_{\Gamma_n}$. It is then straightforward to check $\overline{D}_* \to E_* \to \overline{D}_*$ is the identity, and that the composition $E_* \to \overline{D}_* \to E_* \twoheadrightarrow \overline{E}_*$ agrees with the projection $E_* \twoheadrightarrow \overline{E}_*$. Together these yield the isomorphism $H_*(\overline{D}) \cong H_*(E_*)$. //

Remark 5.11 The surjection $D_* \twoheadrightarrow \overline{D}_*$ in general yields neither a surjection nor injection upon passage to homology; in fact it is not hard to work out an explicit description of the homology groups themselves which indicates the difference. From the exactness of the complex $\{\Gamma^{n-2}_{n-1}, \partial_{n-1}\}_{n\geq 1}$ we see that $H_1(D_*) = H_1(\overline{D}_*)$, while for $n \geq 2$ one has

$$H_n(D_*) = \frac{\Gamma^{n-2}_{n-2} \cap [\Gamma^{n-3}_{n-2}, \Gamma^{n-3}_{n-2}]}{[\Gamma^{n-2}_{n-2}, \Gamma^{n-2}_{n-2}]}$$
$$H_n(\overline{D}_*) = \frac{\Gamma^{n-2}_{n-2} \cap [\Gamma^{n-3}_{n-2}, \Gamma_{n-2}]}{[\Gamma^{n-2}_{n-2}, \Gamma_{n-2}]}$$

These groups are the same for $n = 2$ but will differ in general when $n > 2$. When $n = 2$, the expression for $H_2(D_*) = H_2(\overline{D}_*)$ is precisely Hopf's formula for $H_2(B\pi)$, so the description of $H_n(\overline{D}_*)$ when $n > 2$ may be seen as a higher dimensional analogue of this formula. The equalities can be formulated integrally, and hold integrally.

By Lemma 5.10, we see that any $0 \neq x \in H_n(B\pi;\mathbb{Q}) = H_n(\overline{D}_*)$ maps non-trivially to $\overline{D}_n/im(d^{\overline{D}}_{n+1})$, and in turn can be lifted (non-uniquely) to an element $0 \neq y_x \in D_n/im(d^D_{n+1})$ under the surjection $D_n/im(d^D_{n+1}) \twoheadrightarrow \overline{D}_n/im(d^{\overline{D}}_{n+1})$. Consider

the following commutative diagrams (in which $A \otimes \mathbb{Q}$ is abbreviated by $A^{\mathbb{Q}}$).

$$
\begin{tikzcd}
 & & H_n(B\pi;\mathbb{Q}) \arrow[d, tail] \\
H_1(B\Gamma_n^{n-1};\mathbb{Q})_{\Gamma_n} \arrow[r, "(\overline{\partial}_n)_*"] & H_1(B\Gamma_{n-1}^{n-2};\mathbb{Q})_{\Gamma_{n-1}} \arrow[r] & coker_1^{\overline{D},n} \\
H_1(B\Gamma_n^{n-1};\mathbb{Q}) \arrow[u, two heads] \arrow[r, "(\partial_n)_*"] \arrow[d] & H_1(B\Gamma_{n-1}^{n-2};\mathbb{Q}) \arrow[u, two heads] \arrow[r] \arrow[d] & coker_1^{D,n} \arrow[u, two heads] \arrow[d] \\
K_1^a(\partial_n(I[\Gamma_n^{n-1}]))^{\mathbb{Q}} \arrow[r] \arrow[d] & K_1^a(I[\Gamma_{n-1}^{n-2}])^{\mathbb{Q}} \arrow[r] \arrow[d] & coker_2^n \arrow[d] \\
K_1^a(\partial_n(C^i(\Gamma.^+)_n^{n-1}))^{\mathbb{Q}} \arrow[r] \arrow[d] & K_1^a(C^i(\Gamma.^+)_{n-1}^{n-2})^{\mathbb{Q}} \arrow[r] \arrow[d] & coker_3^{a,n} \arrow[d] \\
K_1^t(\partial_n(C^i(\Gamma.^+)_n^{n-1}))^{\mathbb{Q}} \arrow[r] & K_1^t(C^i(\Gamma.^+)_{n-1}^{n-2})^{\mathbb{Q}} \arrow[r] & coker_3^{t,n}
\end{tikzcd}
\tag{5.12}
$$

Here $I[G] = ker(\mathbb{Z}[G] \to \mathbb{Z})$ and $H_1(B\Gamma_{n-1}^{n-2};\mathbb{Q}) \to K_1^a(I[\Gamma_{n-1}^{n-2}])^{\mathbb{Q}}$ is the (reduced, restricted) assembly map for Γ_{n-1}^{n-2} in algebraic K-theory. The map $H_1(B\Gamma_n^{n-1};\mathbb{Q}) \to K_1^a(\partial_n(I[\Gamma_n^{n-1}]))^{\mathbb{Q}}$ is the composition $H_1(B\Gamma_n^{n-1};\mathbb{Q}) \to K_1^a(I[\Gamma_n^{n-1}])^{\mathbb{Q}} \to K_1^a(\partial_n(I[\Gamma_n^{n-1}]))^{\mathbb{Q}}$. The vertical maps from the fourth to the fifth lines are induced by the natural transformation $\mathbb{Z}[\ ] \to C^i(\ )$; from the fifth to the sixth lines by the transformation from algebraic to topological K-theory. Finally, the groups on the right are the cokernels of the respective maps on each line, while the vertical maps are those induced by maps of cokernels.

Let $[c] \in H^n(B\pi;\mathbb{Q})$. As noted above in (5.5), $[c]$ identifies with a simplicial homomorphism $\phi_c : \Gamma.^+ \to K(\mathbb{Q}, n-1).$; in turn ϕ_c induces a simplicial algebra homomorphism $C^i(\Gamma.^+) \to C^i(K(\mathbb{Q}, n-1).)$ which we also denote be ϕ_c. Following

(5.9) our second diagram is

(5.13)

$$\begin{tikzcd}
H_1(B\Gamma_n^{n-1};\mathbb{Q}) \arrow[r, "(\partial_n)_*"] \arrow[d] & H_1(B\Gamma_{n-1}^{n-2};\mathbb{Q}) \arrow[r] \arrow[d] & coker_1^{D,n} \arrow[d] \\
K_1^a(\partial_n(C^i(\Gamma.^+)_n^{n-1}))^{\mathbb{Q}} \arrow[r] \arrow[d] & K_1^a(C^i(\Gamma.^+)_{n-1}^{n-2})^{\mathbb{Q}} \arrow[r] \arrow[d] & coker_3^{a,n} \arrow[d] \\
K_1^a(\partial_n(C^i(K(\mathbb{Q},n-1).)_n^{n-1}))^{\mathbb{Q}} \arrow[r] \arrow[d, equal] & K_1^a(C^i(K(\mathbb{Q},n-1).)_{n-1}^{n-2})^{\mathbb{Q}} \arrow[r] \arrow[d, equal] & coker_4^{a,n} \arrow[d] \\
K_1^a(I^i(\mathbb{Q})^2)^{\mathbb{Q}} \arrow[r] & K_1^a(I^i(\mathbb{Q}))^{\mathbb{Q}} \arrow[r] & K_1^a(I^i(\mathbb{Q})/I^i(\mathbb{Q})^2)^{\mathbb{Q}} \arrow[d, "\cong"] \\
 & & \mathbb{C}
\end{tikzcd}$$

Given $0 \neq x \in H_n(B\pi;\mathbb{Q})$, we choose a lift $y_x \in coker_1^{D,n}$ which maps to the image of x in $coker_1^{\overline{D},n}$. Mapping y_x by the composition on the right-hand side of (5.13) yields an element in $\mathbb{C}$ equal to the image of $< [c], x > \in \mathbb{Q}$ under the inclusion $\mathbb{Q} \hookrightarrow \mathbb{C}$. In particular, it depends only on x and not the choice of lift y_x. Referring to these diagrams, we may summarize this as

Theorem 5.14. *The Strong Novikov Conjecture is equivalent to the statement:*

(i) For all $n \geq 1$ and $y \in coker_1^{D,n}$, if y projects to a non-zero element in $im(H_n(B\pi;\mathbb{Q}) \rightarrowtail coker_1^{\overline{D},n})$, then it maps to a non-zero element in $coker_3^{t,n}$.

Moreover, the algebraic analogue of (i) is always true:

(ii) For all $n \geq 1$ and $y \in coker_1^{D,n}$, if y projects to a non-zero element in $im(H_n(B\pi;\mathbb{Q}) \rightarrowtail coker_1^{\overline{D},n})$, then it maps to a non-zero element in $coker_3^{a,n}$.

Proof. The first part is a restatement of Theorem 5.4, in the context of (5.12). The second part follows from the Universal Coefficient Theorem. //

This method of detecting the image of the restricted assembly map does not work for topological K-theory directly, as the inclusion $(I^i(\mathbb{Q}))^2 \hookrightarrow I^i(\mathbb{Q})$ induces an isomorphism of topological K-groups.

Theorem 5.14 can be restated in terms of absolute K-groups. Let $\widetilde{C}_{n-1}^i(\Gamma.^+) = C^i(\Gamma.^+)_{n-1}^{n-2}/\partial_n(C^i(\Gamma.^+)_n^{n-1})$ equipped with the fine topology.

Theorem 5.15. *The Strong Novikov Conjecture is equivalent to the statement:*

(i) For all $n \geq 1$ and $y \in coker_1^{D,n}$, if y projects to a non-zero element in $im(H_n(B\pi;\mathbb{Q}) \rightarrowtail coker_1^{\overline{D},n})$, then it maps to a non-zero element in $K_1^t(\widetilde{C}^i_{n-1}(\Gamma.^+))^{\mathbb{Q}}$.

Moreover, the algebraic analogue of (i) is always true:

(ii) For all $n \geq 1$ and $y \in coker_1^{D,n}$, if y projects to a non-zero element in $im(H_n(B\pi;\mathbb{Q}) \rightarrowtail coker_1^{\overline{D},n})$, then it maps to a non-zero element in $K_1^a(\widetilde{C}^i_{n-1}(\Gamma.^+))^{\mathbb{Q}}$.

These methods also apply to the rational Witt groups of the group algebra. The augmented simplicial group algebra $\mathbb{Z}[\Gamma.^+]$ is a resolution of $\mathbb{Z}[\pi]$ if $\Gamma.^+$ is a resolution of π, yielding a short-exact sequence of rings-with-involution

$$\mathbb{Z}[\Gamma.^+]^n_n \rightarrowtail \mathbb{Z}[\Gamma.^+]^{n-1}_n \twoheadrightarrow \mathbb{Z}[\Gamma.^+]^{n-1}_{n-1} \qquad n \geq 0$$

As $W_*(\_)^{\mathbb{Q}}$ satisfies excision, there are boundary maps $\partial_* : W_m(\mathbb{Z}[\Gamma.^+]^{n-1}_{n-1})^{\mathbb{Q}} \to W_{m-1}(\mathbb{Z}[\Gamma.^+]^n_n)^{\mathbb{Q}}$. Composing them yields a map of Witt groups

$$\partial^{(n-1)} : W_n(\mathbb{Z}[\pi])^{\mathbb{Q}} \to W_1(\mathbb{Z}[\pi]^{n-2}_{n-2})^{\mathbb{Q}}$$

for each $n \geq 1$. Using the periodicity of the rationalized Witt groups and fact that the assembly map $H_*(BG;\mathbb{Q}) \otimes W_*(\mathbb{Z})^{\mathbb{Q}} \to W_*(\mathbb{Z}[G])^{\mathbb{Q}}$ is an isomorphism when G is a free group, the same line of argument as above implies

Theorem 5.16. *The Novikov conjecture is true for π if and only if the composition*

$$H_n(B\pi;\mathbb{Q}) \to W_n(\mathbb{Z}[\pi])^{\mathbb{Q}} \xrightarrow{\partial^{(n-1)}} W_1(\mathbb{Z}[\pi]^{n-2}_{n-2})^{\mathbb{Q}}$$

is injective for each $n \geq 1$.

The first map in this composition is induced by the inclusion of the summand $H_*(B\pi;\mathbb{Q}) \rightarrowtail H_*(BG;\mathbb{Q}) \otimes W_*(\mathbb{Z})^{\mathbb{Q}}$, followed by the assembly map to $W_*(\mathbb{Z}[G])^{\mathbb{Q}}$.

6 Some additional remarks

We conclude by describing some additional properties of C^i-algebras. As we have noted, $C^i(\_)$ is functorial with respect to group homomorphisms, hence extends to

a functor from $(s.gps)$ (the category of simplicial groups) to $(s.t.alg.)$ (the category of simplicial topological algebras). Recall also that a simplicial set $X.$ is said to be r-reduced if $X_k = \{pt\}$ for all $k \leq r$.

Lemma 6.1. *Let $G.$ be a 0-reduced simplicial group. Then the inclusion of simplicial algebras $\mathbb{C}[G.] \hookrightarrow C^i(G.)$ is a weak homotopy equivalence (the homotopy groups being the simplicial homotopy groups as defined in section 1).*

Proof. Consider the commuting diagram

$$\begin{array}{ccc} \mathbb{C}[G.] & \longrightarrow & \mathbb{C}[G.]\hat{} \\ \downarrow & & \downarrow \\ C^i(G.) & \longrightarrow & C^i(G.)\hat{} \end{array}$$

where the horizontal arrows denote completion with respect to the (simplicial) augmentation ideal, while the vertical arrows arise from the natural transformation $\mathbb{C}[\_] \hookrightarrow C^i(\_)$. By Curtis convergence [Cu], the horizontal maps are weak equivalences. By Lemma 5.7, the right vertical map is an isomorphism. The result follows. //

For a simplicial algebra $A.$ over $\mathbb{C}$, $C_*(A.)$ resp. $CC_*(A.)$ will denote the total Hochschild resp. cyclic complex associated to the simplicial complexes $\{[n] \mapsto C_*(A_n)\}_{n\geq 0}$ resp. $\{[n] \mapsto CC_*(A_n)\}_{n\geq 0}$. For algebras topologized by the fine topology, the algebraic and topological complexes are the same, so there is no need to distinguish between the two.

Question 6.2 Is $C^i(\Gamma.)$ a resolution of $\widetilde{C}^i(\Gamma.) \overset{def}{=} \pi_0(C^i(\Gamma.))$?

Question 6.3 Does the image of $H_*(B\pi;\mathbb{Q}) \to HH_*(\mathbb{C}[\pi]) \to HH_*(\widetilde{C}^i(\Gamma.))$ map injectively under $I : HH_*(\widetilde{C}^i(\Gamma.)) \to HC_*(\widetilde{C}^i(\Gamma.))$?

Theorem 6.4. *If the answer to these two questions is "yes" for a given resolution $\Gamma.$ of π, then the Strong Novikov Conjecture is true for π.*

Proof. If $C^i(\Gamma.)$ is a resolution of $\widetilde{C}^i(\Gamma.)$, then the augmentation $C^i(\Gamma.) \twoheadrightarrow \widetilde{C}^i(\Gamma.)$ induces a quasi-isomorphism of chain complexes $C_*(C^i(\Gamma.)) \overset{\simeq}{\twoheadrightarrow} C_*(\widetilde{C}^i(\Gamma.))$. Given

$0 \neq x \in H_n(B\pi; \mathbb{Q})$, choose a cohomology class $[\phi] \in H^n(B\pi; \mathbb{Q})$ which pairs non-trivially with x, and then a simplicial group homomorphism $\phi : \Gamma. \to K(\mathbb{Q}, n-1).$ representing $[\phi]$ in the manner discussed above. We may assume without loss of generality that $n > 1$, as injectivity in the case $n = 1$ may be verified directly. Now consideration of the commuting diagram

$$\begin{array}{ccccc}
H_*(B\Gamma.;\mathbb{Q}) & \longrightarrow & HH_*(C^i(\Gamma.)) & \xrightarrow{(\phi)_*} & HH_*(C^i(K(\mathbb{Q}, n-1).)) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
H_*(B\pi;\mathbb{Q}) & \longrightarrow & HH_*(\widetilde{C}^i(\Gamma.)) & & HH_*(\mathbb{C}[K(\mathbb{Q}, n-1).])
\end{array}$$

verifies injectivity of the composition $H_*(B\pi; \mathbb{Q}) \to HH_*(\mathbb{C}[\pi]) \to HH_*(\widetilde{C}^i(\Gamma.))$. If the composition of this map with $I : HH_*(\widetilde{C}^i(\Gamma.)) \to HC_*(\widetilde{C}^i(\Gamma.))$ is still injective, as stated in (6.3), then the restricted assembly map

$$H_*(B\pi; \mathbb{Q}) \to K^t_*(\widetilde{C}^i(\Gamma.))$$

is injective, via the Chern-Connes-Karoubi character defined in [O1, App.]. Finally, by a five lemma argument one verifies that the natural map $\widetilde{C}^i(\Gamma.) \to C^*(\pi)$ induces an isomorphism of topological K-groups in dimensions ≥ 1. By what we have previously shown, this implies the Strong Novikov Conjecture for π. //

It is not hard to show that given two free simplicial resolutions $\Gamma.$, $\Gamma.'$ of π, the resulting simplicial algebras $C^i(\Gamma.)$ and $C^i(\Gamma.')$ are homotopy equivalent. Thus the homotopy groups of $C^i(\Gamma.)$ are invariants of π, and either the answer to (6.2) is "yes"or one has a new set of invariants associated to the discrete group π.

References

[B] P. Baum, *(private communication).*

[BC] P. Baum and A. Connes, *Chern character for discrete groups*, A fête of topology (1988), 163 – 232.

[BCH] P. Baum, A. Connes and N. Higson, *Classifying space for proper actions and K-theory of group C^*-algebras*, Cont. Math. **167** (1994), 241 – 291.

[BHM] M. Bökstedt, W. C. Hsiang, I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111 (3)** (1993), 465 – 539.

[C1] A. Connes, *Non-Commutative Differential Geometry*, Publ. Math. I.H.E.S. **62** (1985), 41–144.

[C2] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[CM] A. Connes and H. Moscovici, *Hyperbolic groups and the Novikov conjecture*, Topology **29** (1990), 345 – 388.

[CGM] A. Connes, M. Gromov and H. Moscovici, *Group cohomology with Lipschitz control and higher signatures*, Geom. Funct. Anal. **3** (1993), 1 – 78.

[Cu] E. B. Curtis, *Some relations between homotopy and homology*, Ann. of Math. **83** (1965), 386 – 413.

[DL] J. Davis and W. Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K and L-theory*, K-theory **15 (3)** (1998), 201 – 252.

[Di] J. Dixmier, *C^*-algebras*, North Holland, 1977.

[FJ] T. Farrell and L. Jones, *isomorphism conjectures in algebraic K-theory*, Jour. Amer. math. Soc. **6** (1993), 249 – 297.

[HR] N. Higson and J. Roe, *On the course Baum-Connes conjecture*, London Math. Soc. Lect. Note Ser. **227** (1995), 227 – 254.

[J1] P. Jolissaint, *K-Theory of Reduced C^*-Algebras and Rapidly Decreasing Functions on Groups*, K-Theory **2** (1989), 723 – 735.

[K] M. Karoubi, *K-theory. An introduction,* Grundlehren der Mathematischen Wissenschaften, Band 226, Springer-Verlag, Berlin-New York, 1978.

[Ka] G.G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147–210.

[KS] G. Kasparov and G. Skandalis, *Groups acting properly on "bolic"spaces and the Novikov conjecture*, Ann. of Math. (2) **158 (1)** (2003), 165 – 206.

[Mi] A. Mishchenko, *Infinite-dimensional representations of discrete groups and higher signatures*, Izv. Akad. Nauk. SSSR Ser. Mat. **38** (1974), 81 – 106.

[GV] G. Mislin and A. Valette, *Proper group actions and the Baum-Connes Conjecture*, Adv. Courses in Math., CRM Barcelona (2003).

[N] S. P. Novikov, *Homotopic and topological invariance of certain rational classes on Pontrjagin*, Dokl. Akad. Nauk. SSSR **162** (1965), 1248 – 1251.

[O1] C. Ogle, *P-bounded cohomology and absolutely summable algebras*, (submitted).

[Sch] L. Scweitzer, *A short proof that $M_n(A)$ is local if A is local and Fréchet*, Internat. J. Math. **3** (1992), 581 – 589.

[Y2] G. Yu, *The coarse Baum-Connes conjecture for groups with finite asymtotic dimension*, Ann. of Math. (2) **147** (1998), 325 – 355.

[Y1] G. Yu, *Course Baum-Connes conjecture*, K-theory **9 (3)** (1995), 199 – 221.

Dept. of Mathematics, The Ohio State University

E-mail address: ogle@math.ohio-state.edu